Galois theory for Hopf algebroids

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Abstract

An extension $B \subset A$ of algebras over a commutative ring k is an $\mathcal{H}$-extension for an L-bialgebroid $\mathcal{H}$ if A is an $\mathcal{H}$-comodule algebra and B is the subalgebra of its coinvariants. It is $\mathcal{H}$-Galois if in addition the canonical map $A \otimes_B A \to A \otimes_L \mathcal{H}$ is an isomorphism or, equivalently, if the canonical coring $(A \otimes_L \mathcal{H} : A)$ is a Galois coring.

In the case of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, a comodule algebra A is defined as an algebra carrying compatible comodule structures over both constituent bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$. If the antipode is bijective then A is proven to be an $\mathcal{H}_R$-Galois extension of its coinvariants if and only if it is an $\mathcal{H}_L$-Galois extension.

Results about bijective entwining structures are extended to entwining structures over non-commutative algebras in order to prove a Kreimer-Takeuchi type theorem for a finitely generated projective Hopf algebroid $\mathcal{H}$ with a bijective antipode. It states that any $\mathcal{H}$-Galois extension $B \subset A$ is projective, and if A is k-flat then already the surjectivity of the canonical map implies the Galois property.

The Morita theory, developed for corings by Caenepeel, Vercruysse and Wang, is applied to obtain equivalent criteria for the Galois property of Hopf algebroid extensions. This leads to Hopf algebroid analogues of results for Hopf algebra extensions by Doi and, in the case of Frobenius Hopf algebroids, by Cohen, Fishman and Montgomery.

1 Introduction

An extension $B \subset A$ of algebras over a commutative ring k is an H-extension for a k-bialgebra H if A is a right H-comodule algebra and B is the subalgebra of its coinvariants i.e. of elements $b \in A$ such that $b_{\langle 0\rangle} \otimes b_{\langle 1\rangle} = b \otimes 1_H$ – where the map $A \to A \otimes_k H$, $a \mapsto a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ is the coaction of H on A (summation understood). An H-extension $B \subset A$ is H-Galois if the canonical map $A \otimes_B A \to A \otimes_k H$, $a \otimes a' \mapsto aa'_{\langle 0\rangle} \otimes a'_{\langle 1\rangle}$ is an isomorphism of k-modules.

In many cases it is technically much easier to check the surjectivity of the canonical map than its injectivity. A powerful tool in the study of H-extensions is the Kreimer-Takeuchi theorem [27] stating that if H is a finitely generated projective Hopf algebra then the surjectivity of the canonical map implies its bijectivity and also the fact that A is projective both as a left and as a right B-module.

The proof of the Kreimer-Takeuchi theorem went through both simplification and generalization in the papers [32, 34, 11, 33]. In the present paper we adopt the method of Brzeziński [11] and of Schauenburg and Schneider [33], who used the following observation. A comodule algebra A for a bialgebra H determines a canonical entwining structure [13] consisting of the algebra A, the coalgebra underlying the bialgebra H, and the entwining map $H \otimes_k A \to A \otimes_k H$, $h \otimes a \mapsto a_{\langle 0\rangle} \otimes ha_{\langle 1\rangle}$. In the case when the bialgebra H possesses a skew antipode, this entwining map is a bijection. The proof of (a wide generalization of) the Kreimer-Takeuchi theorem both in [11] and in [33] is based on the study of bijective entwining structures, under slightly different assumptions. In Section 4 below we show that these arguments can be repeated almost without modification by using entwining structures over non-commutative algebras [5].

In the paper [23] Doi constructed a Morita context for an H-extension $B \subset A$. If H is finitely generated and projective as a k-module then the surjectivity of one of the connecting maps is equivalent to the projectivity and the Galois property of the extension $B \subset A$, while the strictness of the Morita context is equivalent to faithful flatness and the Galois property. This observation made it possible to use all results of Morita theory for characterizing H-Galois extensions. In the case when H is a finite dimensional Hopf algebra over a field (or a Frobenius Hopf algebra over a commutative ring), the Morita context of Doi is equivalent to another Morita context, introduced by Cohen, Fishman and Montgomery [22].

One of the most beautiful applications of the theory of corings [16] is the observation [10] that the Galois property of an H-extension $B \subset A$ is equivalent to the Galois property of a canonical A-coring $A \otimes H$. In [20] the construction of the Morita context by Doi has been extended to any A-coring C possessing a grouplike element (i.e. such that A is a C-comodule). In the case when C is a finitely generated projective A-module (or an A-progenerator, see [17]) the application of Morita theory yields then several equivalent criteria for the Galois property of the coring C and the projectivity (or faithful flatness) of A as a module for the subalgebra of coinvariants in A. In the case when the A-dual algebra of the coring C is a Frobenius extension of A, also the Morita context in [22] has been generalized to the general setting of corings and the precise relation of the two Morita contexts has been explained.

The notion of bialgebra extensions has been generalized to bialgebroids by Kadison [25] as an extension $B \subset A$ of k-algebras such that A is a comodule algebra and B is the subalgebra of coinvariants. The Galois property of a bialgebroid extension can be formulated also as the Galois property of a canonical coring. This implies that the general theory, developed in [20], can be applied also to bialgebroid extensions.

In the present paper we study Hopf algebroid extensions. The notion of Hopf algebroids has been introduced in [9, 4] and studied further in [6]. It consists of two compatible (left and right) bialgebroid structures on the same algebra which are related by the antipode.

In Section 3 a comodule of a Hopf algebroid is defined as a pair of comodules for both constituent bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$, in such a way that the $\mathcal{H}_R$-coaction is $\mathcal{H}_L$-colinear and the $\mathcal{H}_L$-coaction is $\mathcal{H}_R$-colinear. In particular, a comodule algebra A of a Hopf algebroid is defined as an algebra carrying the structure of a compatible pair of comodule algebras of the constituent bialgebroids. If the antipode of a Hopf algebroid $\mathcal{H}$ is bijective, then we prove that the $\mathcal{H}_R$- and the $\mathcal{H}_L$-coinvariants of any $\mathcal{H}$-comodule algebra A coincide. What is more, we show that in this case A is a Galois extension of its coinvariant subalgebra by $\mathcal{H}_L$ if and only if it is a Galois extension by $\mathcal{H}_R$.

In Section 4 it is shown that – just as in the case of Hopf algebras – if $\mathcal{H}$ is a Hopf algebroid with a bijective antipode then the canonical entwining structure (over the non-commutative base algebra of $\mathcal{H}$), associated to an $\mathcal{H}$-comodule algebra, is bijective. This fact is used to prove a Kreimer-Takeuchi type theorem.

In Section 5 we apply the Morita theory for corings to a Hopf algebroid extension $B \subset A$, looked at as an extension by the constituent right bialgebroid $\mathcal{H}_R$. In the finitely generated projective case this results in equivalent criteria, under which $B \subset A$ is a projective $\mathcal{H}_R$-Galois extension. Similarly, we can look at $B \subset A$ as an extension by the constituent left bialgebroid $\mathcal{H}_L$ and obtain equivalent conditions for its projectivity and $\mathcal{H}_L$ -Galois property. Making use of the results about Hopf algebroid extensions in Section 3, and the Kreimer-Takeuchi type theorem proven in Section 4, we conclude that if $\mathcal{H}$ is a finitely generated projective Hopf algebroid with a bijective antipode then the two equivalent sets of conditions are equivalent also to each other. In the case of Frobenius Hopf algebroids [6] we obtain a direct generalization of ([22], Theorem 1.2).

Throughout the paper k is a commutative ring. By an algebra $R = (R, \mu, \eta)$ we mean an associative unital k-algebra. Instead of the unit map η we use sometimes the unit element $1_R := \eta(1_k)$. We denote by ${}_R\mathcal{M}$, $\mathcal{M}_R$ and ${}_R\mathcal{M}_R$ the categories of left, right, and bimodules for R, respectively. For the k-module of morphisms in ${}_R\mathcal{M}$, $\mathcal{M}_R$ and ${}_R\mathcal{M}_R$ we write ${}_R\mathrm{Hom}(\ ,\ )$, $\mathrm{Hom}_R(\ ,\ )$ and ${}_R\mathrm{Hom}_R(\ ,\ )$, respectively.

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2 Preliminaries

2.1 Bialgebroids and Hopf algebroids

L -bialgebroids [28, 39, 36, 30] or, what were shown in [14] to be equivalent to them, $\times_L$ -bialgebras [38] are generalizations of bialgebras to the case of non-commutative base algebras. This means that instead of coalgebras and algebras over commutative rings, one works with corings and rings over non-commutative base algebras. Recall that a coring over a k-algebra L is a comonoid in ${}_L\mathcal{M}_L$ while an L-ring is a monoid in ${}_L\mathcal{M}_L$. The notion of L-rings is equivalent to a pair, consisting of a k-algebra A and an algebra homomorphism $L \to A$.

Definition 2.1 A *left bialgebroid* is a 6-tuple $\mathcal{H} = (H, L, s, t, \gamma, \pi)$, where H and L are k-algebras. H is an $L \underset{k}{\otimes} L^{op}$-ring via the algebra homomorphisms $s : L \to H$ and $t : L^{op} \to H$, the images of which are required to commute in H. In terms of the maps s and t one equips H with an L-L bimodule structure as

$$l \cdot h \cdot l' :\, = s(l)t(l')h \qquad \text{for } h \in H,\ l, l' \in L. \tag{2.1}$$

The triple (H, γ, π) is an L-coring with respect to the bimodule structure (2.1). Introducing Sweedler's notation $\gamma(h) = h_{(1)} \underset{L}{\otimes} h_{(2)}$ for $h \in H$ (with implicit summation understood), the axioms

$$h_{(1)}t(l) \underset{L}{\otimes} h_{(2)} = h_{(1)} \underset{L}{\otimes} h_{(2)}s(l) \tag{2.2}$$
$$\gamma(1_H) = 1_H \underset{L}{\otimes} 1_H \tag{2.3}$$
$$\gamma(hh') = \gamma(h)\gamma(h') \tag{2.4}$$
$$\pi(1_H) = 1_L \tag{2.5}$$
$$\pi\left(h\ s{\circ}\pi(h')\right) = \pi(hh') = \pi\left(h\ t{\circ}\pi(h')\right) \tag{2.6}$$

are required for all $l \in L$ and $h, h' \in H$.

Notice that – although $H \underset{L}{\otimes} H$ is not an algebra – axiom (2.4) makes sense in view of (2.2).

The bimodule (2.1) is defined in terms of multiplication by s and t on the left. The R-R bimodule structure in a *right bialgebroid* $\mathcal{H} = (H, R, s, t, \gamma, \pi)$ is defined in terms of multiplication on the right. For the details we refer to [26].

The *opposite* of a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$ is the right bialgebroid $\mathcal{H}^{op} = (H^{op}, L, t, s, \gamma, \pi)$ where H^{op} is the algebra opposite to H. The *co-opposite* of $\mathcal{H}$ is the left bialgebroid $\mathcal{H}_{cop} = (H, L^{op}, t, s, \gamma^{op}, \pi)$ where $\gamma^{op} : H \to H \underset{L^{op}}{\otimes} H$ is the opposite coproduct $h \mapsto h_{(2)} \underset{L^{op}}{\otimes} h_{(1)}$.

It has been observed in [26] that for a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$, such that H is finitely generated and projective as a left or right L-module, the corresponding L-*dual* possesses a right bialgebroid structure over the base algebra L. Analogously, the R-duals of a finitely generated projective right bialgebroid over R possess left bialgebroid structures. For the explicit forms of the dual bialgebroid structures consult [26].

Before defining the notion of a Hopf algebroid, let us introduce some notations. Analogous notations were used already in [9, 6].

When dealing with an $L \underset{k}{\otimes} L^{op}$-ring H, we have to face the situation that H carries different module structures over the base algebra L. In this situation the usual notation $H \underset{L}{\otimes} H$ would be ambiguous. Therefore we make the following notational convention. In terms of the algebra homomorphisms $s : L \to H$ and $t : L^{op} \to H$ (with commuting images in H) we introduce four

L-modules

$$
\begin{aligned}
{}_L H: &\qquad l \cdot h := s(l)h \\
H_L: &\qquad h \cdot l := t(l)h \\
H^L: &\qquad h \cdot l = hs(l) \\
{}^L H: &\qquad l \cdot h = ht(l).
\end{aligned} \tag{2.7}
$$

This convention can be memorized as left indices stand for left modules and right indices stand for right modules. Upper indices refer to modules defined in terms of right multiplication and lower indices refer to the ones defined in terms of left multiplication.

In writing L-module tensor products, we write out explicitly the module structures of the factors that are taking part in the tensor products, and do not put marks under the symbol $\otimes$. E.g. we write $H_L \otimes {}_L H$. In writing elements of tensor product modules we do not distinguish between the various module tensor products. That is, we write both $h \underset{L}{\otimes} h' \in H_L \otimes {}_L H$ and $g \underset{L}{\otimes} g' \in H^L \otimes {}_L H$, for example.

A left L-module can be considered canonically as a right L^{op}-module, and sometimes we want to take a module tensor product over L^{op}. In this case we use the name of the corresponding L-module and the fact that the tensor product is taken over L^{op} should be clear from the order of the factors.

In writing multiple tensor products we use different types of letters to denote which module structures take part in the same tensor product.

Definition 2.2 A *Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ consists of a left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$ and a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, with common total algebra H, and a k-module map $S : H \to H$, called the antipode, such that the following axioms hold true [1]:

$$
\begin{aligned}
&i) \qquad s_L \circ \pi_L \circ t_R = t_R, \qquad t_L \circ \pi_L \circ s_R = s_R \quad \text{and} \\
&\qquad\;\; s_R \circ \pi_R \circ t_L = t_L, \qquad t_R \circ \pi_R \circ s_L = s_L;
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
&ii) \qquad (\gamma_L \otimes {}^R H) \circ \gamma_R = (H_L \otimes \gamma_R) \circ \gamma_L \quad \text{as maps } H \to H_L \otimes {}_L H^R \otimes {}^R H \quad \text{and} \\
&\qquad\;\; (\gamma_R \otimes {}_L H) \circ \gamma_L = (H^R \otimes \gamma_L) \circ \gamma_R \quad \text{as maps } H \to H^R \otimes {}^R H_L \otimes {}_L H;
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
&iii) \qquad S \text{ is both an } L\text{-}L \text{ bimodule map } {}^L H_L \to {}_L H^L \text{ and an } R\text{-}R \text{ bimodule map} \\
&\qquad\;\; {}^R H_R \to {}_R H^R;
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&iv) \qquad \mu_H \circ (S \otimes {}_L H) \circ \gamma_L = s_R \circ \pi_R \quad \text{and} \\
&\qquad\;\; \mu_H \circ (H^R \otimes S) \circ \gamma_R = s_L \circ \pi_L.
\end{aligned} \tag{2.11}
$$

If $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a Hopf algebroid then so is $\mathcal{H}^{op}_{cop} = ((\mathcal{H}_R)^{op}_{cop}, (\mathcal{H}_L)^{op}_{cop}, S)$ and if S is bijective then also $\mathcal{H}_{cop} = ((\mathcal{H}_L)_{cop}, (\mathcal{H}_R)_{cop}, S^{-1})$ and $\mathcal{H}^{op} = ((\mathcal{H}_R)^{op}, (\mathcal{H}_L)^{op}, S^{-1})$.

We are going to use the following variant of the Sweedler-Heynemann index convention. For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ we use the notation $\gamma_L(h) = h_{(1)} \otimes h_{(2)}$ with lower indices and $\gamma_R(h) = h^{(1)} \otimes h^{(2)}$ with upper indices for $h \in H$ in the case of the coproducts of $\mathcal{H}_L$ and $\mathcal{H}_R$, respectively. In both cases implicit summation is understood. Axioms (2.9) read in this notation as

$$
\begin{aligned}
{h^{(1)}}_{(1)} \otimes {h^{(1)}}_{(2)} \otimes h^{(2)} &= h_{(1)} \otimes {h_{(2)}}^{(1)} \otimes {h_{(2)}}^{(2)} \\
{h_{(1)}}^{(1)} \otimes {h_{(1)}}^{(2)} \otimes h_{(2)} &= h^{(1)} \otimes {h^{(2)}}_{(1)} \otimes {h^{(2)}}_{(2)}
\end{aligned}
$$

for $h \in H$.

It is proven in ([6], Proposition 2.3) that the base algebras L and R of the left and right bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$ in a Hopf algebroid $\mathcal{H}$ are anti-isomorphic via any of the isomorphisms $\pi_L \circ s_R$ and $\pi_L \circ t_R$. The antipode is a homomorphism of left bialgebroids $\mathcal{H}_L \to (\mathcal{H}_R)^{op}_{cop}$ and also $(\mathcal{H}_R)^{op}_{cop} \to \mathcal{H}_L$ in the sense that it is an anti-algebra endomorphism of H and the

[1] An equivalent set of axioms is obtained by requiring S to be only an R-L bimodule map, see Remark 2.1 in [7].

pair of maps $(S, \pi_L \circ s_R)$ is a coring homomorphism from the R^{op}-coring $(H, \gamma_R^{op}, \pi_R)$ to the L-coring (H, γ_L, π_L) and the pair of algebra homomorphisms $(S, \pi_R \circ s_L)$ is a coring homomorphism $(H, \gamma_L, \pi_L) \to (H, \gamma_R^{op}, \pi_R)$.

We term a Hopf algebroid $\mathcal{H}$, for that all modules H^R, ${}^R H$, H_L and ${}_L H$ is finitely generated and projective, as a *finitely generated projective* Hopf algebroid.

Left *integrals* in a left bialgebroid $\mathcal{H}_L$ are defined ([9], Definition 5.1) as the invariants of the left regular H-module i.e. the elements of

$$\mathcal{L}(H) := \{ \ell \in H \mid h\ell = s_L \circ \pi_L(h)\, \ell \quad \forall h \in H \}.$$

By ([6], Scholium 2.8) an element ℓ of a Hopf algebroid $\mathcal{H}$ is a left integral if and only if $h\ell^{(1)} \underset{R}{\otimes} S(\ell^{(2)}) = \ell^{(1)} \underset{R}{\otimes} S(\ell^{(2)})h$ for all $h \in H$.

A left integral ℓ in a Hopf algebroid $\mathcal{H}$ is called *non-degenerate* ([9], Definition 5.3) if both maps

$$\begin{array}{lll} \ell_R : H^* := \mathrm{Hom}_R(H^R, R) \to H & \phi^* \mapsto \phi^* \rightharpoonup \ell \equiv \ell^{(2)}\ t_R \circ \phi^*(\ell^{(1)}) & \text{and} \\ {}_R\ell : {}^*H := {}_R\mathrm{Hom}({}^R H, R) \to H & {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell \equiv \ell^{(1)}\ s_R \circ {}^*\phi(\ell^{(2)}) & \end{array}$$

are isomorphisms. By ([9], Proposition 5.10) for a non-degenerate left integral ℓ in a Hopf algebroid $\mathcal{H}$ also the maps

$$\begin{array}{lll} \ell_L : H_* := \mathrm{Hom}_L(H_L, L) \to H & \phi_* \mapsto \ell \leftharpoonup \phi_* \equiv s_L \circ \phi_*(\ell_{(1)})\ \ell_{(2)} & \text{and} \\ {}_L\ell : {}_*H := {}_L\mathrm{Hom}({}_L H, L) \to H & {}_*\phi \mapsto \ell \leftharpoondown {}_*\phi \equiv t_L \circ {}_*\phi(\ell_{(2)})\ \ell_{(1)} & \end{array}$$

are isomorphisms. It is shown in ([6], Theorem 4.7, see also the *Corrigendum*) that the existence of a non-degenerate left integral in a finitely generated projective Hopf algebroid $\mathcal{H}$ is equivalent to the Frobenius property of any of the four extensions $s_R : R \to H$, $t_R : R^{op} \to H$, $s_L : L \to H$ and $t_L : L^{op} \to H$ and it implies the bijectivity of the antipode. What is more, if the Hopf algebroid $\mathcal{H}$ possesses a non-degenerate left integral then also the four duals H^*, ${}^*H$, H_* and ${}_*H$ possess (anti-) isomorphic Hopf algebroid structures with non-degenerate integrals ([9], Theorem 5.17 and Proposition 5.19). Motivated by these results we term a Hopf algebroid possessing a non-degenerate left integral as a *Frobenius* Hopf algebroid. Recall from [37] that a Frobenius Hopf algebroid is equivalent to a distributive Frobenius double algebra.

2.2 Module and comodule algebras

The category ${}_H\mathcal{M}$ of left modules for the total algebra H of a left bialgebroid $(H, L, s, t, \gamma, \pi)$ is a monoidal category. As a matter of fact, any H-module is an L-L bimodule via s and t. The monoidal product in ${}_H\mathcal{M}$ is the L-module tensor product with H-module structure

$$h \cdot (m \underset{L}{\otimes} n) := h_{(1)} \cdot m \underset{L}{\otimes} h_{(2)} \cdot n \qquad \text{for } h \in H,\ m \underset{L}{\otimes} n \in M \underset{L}{\otimes} N$$

and the monoidal unit is L with H-module structure

$$h \cdot l := \pi(hs(l)) \qquad \text{for } h \in H, l \in L.$$

A left H*-module algebra* is defined as a monoid in the monoidal category ${}_H\mathcal{M}$. A left H -module algebra A is in particular an L-ring via the homomorphism

$$L \to A \qquad l \mapsto l \cdot 1_A \equiv 1_A \cdot l.$$

The *invariants* of A are the elements of

$$A^H := \{ a \in A \mid h \cdot a = s \circ \pi(h) \cdot a \quad \forall h \in H \}.$$

Just in the same way, the category $\mathcal{M}_H$ of right modules for the total algebra H of a right R-bialgebroid is a monoidal category with monoidal product the R-module tensor product and

monoidal unit R. A right H-module algebra is a monoid in $\mathcal{M}_H$. A right module algebra is in particular an R-ring. The invariants are defined analogously to the left case in terms of the counit.

By a comodule for a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$ we mean a comodule for the L-coring (H, γ, π). Recall that the category of left $\mathcal{H}$-comodules is also a monoidal category in the following way. Any left $\mathcal{H}$-comodule (M, τ) can be equipped with a right L-module structure via

$$m \cdot l :\, = \pi(m_{\langle -1\rangle} s(l)) \cdot m_{\langle 0\rangle} \qquad \text{for } m \in M,\ l \in L \tag{2.12}$$

where $m_{\langle -1\rangle} \underset{L}{\otimes} m_{\langle 0\rangle}$ stands for $\tau(m)$ (summation understood). Indeed, (2.12) is the unique right action via which M becomes an L-L bimodule and τ becomes an L-L bimodule map from ${}_L M_L$ to the Takeuchi product $H \times_L M$. Recall from [38] that $H \times_L M$ is the L-L submodule of ${}_L H^L_{\mathbf{L}} \otimes_{\mathbf{L}} M$ the elements $\sum_i h_i \underset{L}{\otimes} m_i$ of which satisfy

$$\sum_i h_i \underset{L}{\otimes} m_i \cdot l = \sum_i h_i t(l) \underset{L}{\otimes} m_i \qquad \text{for } \ l \in L. \tag{2.13}$$

This observation amounts to saying that our definition of left $\mathcal{H}$-comodules is equivalent to ([30], Definition 5.5). Hence, without loss of generality, from now on we can think of a comodule in this latter sense. On the basis of ([30], Definition 5.5) the category ${}^H\mathcal{M}$ of left $\mathcal{H}$ -comodules was shown in ([30], Proposition 5.6) to be monoidal. The monoidal product is the L-module tensor product with comodule structure

$$M \underset{L}{\otimes} N \to H \underset{L}{\otimes} M \underset{L}{\otimes} N \qquad m \underset{L}{\otimes} n \mapsto m_{\langle -1\rangle} n_{\langle -1\rangle} \underset{L}{\otimes} m_{\langle 0\rangle} \underset{L}{\otimes} n_{\langle 0\rangle}$$

and the monoidal unit is L with comodule structure

$$L \to L \underset{L}{\otimes} H \simeq H \qquad l \mapsto s(l).$$

Following ([30], Definition 5.7), a left *comodule algebra* for a left bialgebroid $\mathcal{H}$ is a monoid in the monoidal category ${}^H\mathcal{M}$. Notice that – in view of the equivalence of the two definitions of $\mathcal{H}$ -comodules – this definition of comodule algebras is equivalent to ([12], Definition 3.4).

By similar arguments also the category of right $\mathcal{H}$-comodules – that is of right comodules for the L-coring (H, γ, π) – is monoidal. The monoidal product is the L-module tensor product with coaction

$$M \underset{L}{\otimes} N \to M \underset{L}{\otimes} N \underset{L}{\otimes} H \qquad m \underset{L}{\otimes} n \mapsto m_{\langle 0\rangle} \underset{L}{\otimes} n_{\langle 0\rangle} \underset{L}{\otimes} n_{\langle 1\rangle} m_{\langle 1\rangle}$$

and the monoidal unit is L with comodule structure

$$L \to H \qquad l \mapsto t(l),$$

where the left L-module structure of a right (H, γ, π)-comodule M is defined as $l \cdot m :\, = m_{\langle 0\rangle} \cdot \pi(m_{\langle 1\rangle} s(l))$. Similarly to the case of left comodule algebras we expect the coaction of a right comodule algebra A to be multiplicative – i.e. such that $(aa')_{\langle 0\rangle} \underset{L}{\otimes} (aa')_{\langle 1\rangle} = a_{\langle 0\rangle} a'_{\langle 0\rangle} \underset{L}{\otimes} a_{\langle 1\rangle} a'_{\langle 1\rangle}$ for $a, a' \in A$. Therefore we consider the monoidal category $(\mathcal{M}^H)^{op}$, the monoidal structure of which is the opposite of $\mathcal{M}^H$ (i.e. it comes from the monoidal structure of ${}_{L^{op}}\mathcal{M}_{L^{op}}$). A right $\mathcal{H}$-comodule algebra is defined as a monoid in the category $(\mathcal{M}^H)^{op}$.

Notice that a left $\mathcal{H}$-comodule algebra is in particular an L-ring while a right $\mathcal{H}$-comodule algebra is an L^{op}-ring.

The *coinvariants* of a left (right) $\mathcal{H}$-comodule algebra (A, τ) are the elements of the subalgebra

$$A^{coH} :\, = \{\ a \in A \mid \tau(a) = 1_H \underset{L}{\otimes} a\ \} \quad (\ A^{coH} :\, = \{\ a \in A \mid \tau(a) = a \underset{L}{\otimes} 1_H\ \}\ ).$$

Recall that for a left L-bialgebroid $\mathcal{H}$ the left and right L-duals ${}_*H$ and H_* are rings. There is a faithful functor from the category $\mathcal{M}^H$ of right $\mathcal{H}$-comodules to the category $\mathcal{M}_{{}_*H}$ of right ${}_*H$-modules which is an isomorphism if and only if the module ${}_L H$ is finitely generated and projective. There is a faithful functor also from ${}^H\mathcal{M}$ to $\mathcal{M}_{H_*}$ which is an isomorphism if and only if the module H_L is finitely generated and projective.

Left and right comodules, comodule algebras and their coinvariants for a right bialgebroid are defined analogously.

2.3 Entwining structures over non-commutative algebras

Entwining structures over non-commutative algebras were introduced in [5] as mixed distributive laws in the bicategory of [Algebras, Bimodules, Bimodule maps]. This definition is clearly equivalent to a monad in the bicategory of corings i.e. the bicategory of comonads [35] in the bicategory of [Algebras, Bimodules, Bimodule maps]. Explicitly, we have

Definition 2.3 An *entwining structure* over an algebra R is a triple (A, C, ψ) where $A = (A, \mu, \eta)$ is an R-ring, $C = (C, \Delta, \epsilon)$ is an R-coring and ψ is an R-R bimodule map $C \underset{R}{\otimes} A \to A \underset{R}{\otimes} C$ satisfying

$$\begin{aligned}
&\psi \circ (C \underset{R}{\otimes} \eta) = \eta \underset{R}{\otimes} C\\
&(A \underset{R}{\otimes} \epsilon) \circ \psi = \epsilon \underset{R}{\otimes} A\\
&(\mu \underset{R}{\otimes} C) \circ (A \underset{R}{\otimes} \psi) \circ (\psi \underset{R}{\otimes} A) = \psi \circ (C \underset{R}{\otimes} \mu)\\
&(A \underset{R}{\otimes} \Delta) \circ \psi = (\psi \underset{R}{\otimes} C) \circ (C \underset{R}{\otimes} \psi) \circ (\Delta \underset{R}{\otimes} A).
\end{aligned}$$

An entwining structure is *bijective* if ψ is an isomorphism.

It is shown in ([5], Example 4.5) that an entwining structure (A, C, ψ) over the algebra R determines an A-coring structure on $A \underset{R}{\otimes} C$ with A-A bimodule structure

$$a_1 \cdot (a \underset{R}{\otimes} c) \cdot a_2 = a_1 a \psi(c \underset{R}{\otimes} a_2) \qquad \text{for } a_1, a_2 \in A,\ a \underset{R}{\otimes} c \in A \underset{R}{\otimes} C,$$

coproduct $A \underset{R}{\otimes} \Delta$ and counit $A \underset{R}{\otimes} \epsilon$.

Definition 2.4 A right *entwined module* over an R-entwining structure (A, C, ψ) is a right comodule over the corresponding A-coring $A \underset{R}{\otimes} C$. Explicitly, it is a triple (M, ρ, τ), where (M, ρ) is a right A-module, making M in particular a right R-module. The pair (M, τ) is a right C-comodule such that τ is a right A-module map i.e.

$$\tau \circ \rho = (\rho \underset{R}{\otimes} C) \circ (M \underset{R}{\otimes} \psi) \circ (\tau \underset{R}{\otimes} A).$$

A *morphism* of entwined modules is a morphism of comodules for the A-coring $A \underset{R}{\otimes} C$, that is, an A-linear and C-colinear map. The category of entwined modules will be denoted by $\mathcal{M}_A^C$.

The *coinvariants* of an entwined module are its coinvariants for the A-coring $A \underset{R}{\otimes} C$. If the R-coring C possesses a grouplike element, then this the same as C-coinvariants.

By ([16], 18.13 (2)) the forgetful functor $\mathcal{M}_A^C \to \mathcal{M}_A$ possesses a right adjoint, the functor

$$\_ \underset{R}{\otimes} C : \mathcal{M}_A \to \mathcal{M}_A^C \qquad (\ M, \rho\ ) \mapsto (\ M \underset{R}{\otimes} C\ ,\ (\rho \underset{R}{\otimes} C) \circ (M \underset{R}{\otimes} \psi)\ ,\ M \underset{R}{\otimes} \Delta\ ), \tag{2.14}$$

where the right R-module structure of M comes from its A-module structure. What is more, by the self-duality of the notion of R-entwining structures, also ([16], 32.8 (3)) extends to entwining structures over non-commutative algebras. That is, also the forgetful functor $\mathcal{M}_A^C \to \mathcal{M}^C$ possesses a left adjoint, the functor

$$\_ \underset{R}{\otimes} A : \mathcal{M}^C \to \mathcal{M}_A^C \qquad (\ M, \tau\ ) \mapsto (\ M \underset{R}{\otimes} A\ ,\ (M \underset{R}{\otimes} \mu)\ ,\ (M \underset{R}{\otimes} \psi) \circ (\tau \underset{R}{\otimes} A)\ ). \tag{2.15}$$

This implies, in particular, that both $A \underset{R}{\otimes} C$ and $C \underset{R}{\otimes} A$ are entwined modules. The morphism ψ becomes a morphism of entwined modules.

If the R-coring C possesses a grouplike element e, then also the A-coring $A \underset{R}{\otimes} C$ possesses a grouplike element $1_A \underset{R}{\otimes} e$. Hence A is an entwined module via the right regular A-action and the C-coaction

$$A \to A \underset{R}{\otimes} C \qquad a \mapsto \psi(e \underset{R}{\otimes} a).$$

In this case also the functors

$$(\ \_\ )^{coC} : \mathcal{M}_A^C \to \mathcal{M}_{A^{coC}} \qquad \text{and} \qquad \_ \underset{A^{coC}}{\otimes} A : \mathcal{M}_{A^{coC}} \to \mathcal{M}_A^C \tag{2.16}$$

are adjoints ([16], 28.8). (The entwined module structure of $N \underset{A^{coC}}{\otimes} A$, for a right A^{coC}-module N, is defined via the second tensor factor). The unit and the counit of the adjunction are

$$\eta_N : N \to (N \underset{A^{coC}}{\otimes} A)^{coC} \qquad n \mapsto n \underset{A^{coC}}{\otimes} 1_A \quad \text{and}$$
$$\mu_M : M^{coC} \underset{A^{coC}}{\otimes} A \to M \qquad m \underset{A^{coC}}{\otimes} a \mapsto m \cdot a$$

for any right A^{coC}-module N and entwined module M.

2.4 Morita theory for corings

In the paper [20] a Morita context $(A^{^*C}, {}^*C, A, {}^*C^{^*C}, \nu, \mu)$ has been associated to an A-coring C possessing a grouplike element e. Here the ring ${}^*C = {}_A\mathrm{Hom}(C, A)$ is the left A-dual of the A-coring C with multiplication $(fg)(c) = g\left(c_{(1)} \cdot f(c_{(2)})\right)$. The invariants of a right ${}^*C$-module M are defined with the help of the grouplike element e as the elements of

$$M^{^*C} :\, = \{\, m \in M \mid m \cdot f = m \cdot [\epsilon(\,\_\,)f(e)] \quad \forall f \in {}^*C \,\}.$$

In terms of the grouplike element e, the k-module A can be equipped with a right ${}^*C$-module structure as

$$a \cdot f :\, = f(e \cdot a) \qquad \text{for } a \in A, f \in {}^*C.$$

The ring $A^{^*C}$ is the subring of ${}^*C$-invariants of A i.e.

$$A^{^*C} = \{\, b \in A \mid f(e \cdot b) = bf(e) \quad \forall f \in {}^*C \,\}.$$

A is an $A^{^*C}$ - ${}^*C$ bimodule via

$$b \cdot a \cdot f :\, = bf(e \cdot a) = f(e \cdot ba) \qquad \text{for } b \in A^{^*C}, a \in A, f \in {}^*C.$$

${}^*C^{^*C}$ is the k-module of ${}^*C$-invariants of the right regular ${}^*C$-module i.e.

$${}^*C^{^*C} = \{\, q \in {}^*C \mid f(c_{(1)} \cdot q(c_{(2)})) = f(q(c) \cdot e) \quad \forall f \in {}^*C,\ c \in C \,\}.$$

It is a ${}^*C$ - $A^{^*C}$ bimodule via

$$(f \cdot q \cdot b)(c) :\, = q\left(c_{(1)} \cdot f(c_{(2)})\right) b \qquad \text{for} f \in {}^*C,\ q \in {}^*C^{^*C},\ b \in A^{^*C}, c \in C.$$

The connecting maps ν and μ are given as

$$\nu : A \underset{^*C}{\otimes} {}^*C^{^*C} \to A^{^*C} \qquad a \underset{^*C}{\otimes} q \mapsto q(e \cdot a) \quad \text{and}$$
$$\mu : {}^*C^{^*C} \underset{A^{^*C}}{\otimes} A \to {}^*C \qquad q \underset{A^{^*C}}{\otimes} a \mapsto (\ c \mapsto q(c)a\ ).$$

In [20] the following theorem has been proven.

Theorem 2.5 *Let C be an A-coring possessing a grouplike element e, and let $(A^{^*C}, {}^*C, A, {}^*C^{^*C}, \nu, \mu)$ be the Morita context associated to it.*

(1) *([20], Theorem 3.5). If C is finitely generated and projective as a left A-module (hence the categories $\mathcal{M}^C$ and $\mathcal{M}_{^*C}$ are isomorphic and the C-coinvariants coincide with the ${}^*C$-invariants) then the following assertions are equivalent.*

(a) *The map μ is surjective (and, a fortiori, bijective).*

(b) *The functor $(\,\_\,)^{coC} : \mathcal{M}^C \to \mathcal{M}_{A^{coC}}$ is fully faithful.*

(c) *A is a right ${}^*C$-generator.*

(d) *A is projective as a left A^{coC}-module and the map*

$${}^*C \to {}_{A^{coC}}\mathrm{End}(A) \qquad f \mapsto (\ a \mapsto f(e \cdot a)\ )$$

is an algebra anti-isomorphism.

(e) *A is projective as a left A^{coC}-module and the A-coring C with grouplike element e is a Galois coring.*

(2) *([20], Theorem 2.7). If the algebra extension*

$$A \to {}^*C \qquad a \mapsto (\ c \mapsto \epsilon(c)a\ )$$

is a Frobenius extension with Frobenius system $(\psi, u_i \underset{A}{\otimes} v_i)$ then the Morita context $(A^{*C}, {}^*C, A, {}^*C^{*C}, \nu, \mu)$ is equivalent to the Morita context $(A^{*C}, {}^*C, A, A, \nu', \mu')$ via the isomorphism

$$A \to {}^*C^{*C} \qquad a \mapsto (\ c \mapsto \sum_i v_i[c \cdot au_i(e)]\ ).$$

3 Hopf algebroid extensions

An algebra extension $B \subset A$ is an $\mathcal{H}_R$ -extension for a right bialgebroid $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ if A is a right $\mathcal{H}_R$-comodule algebra and B is the subalgebra of $\mathcal{H}_R$-coinvariants of A. In this situation – denoting the R-coring (H, γ, π) by C – the triple $(A, \mathcal{H}_R, C)$ is a Doi-Koppinen datum over R in the sense of ([12], Definition 3.6). This implies that the R-R bimodule map

$$\psi : H \underset{R}{\otimes} A \to A \underset{R}{\otimes} H \qquad h \underset{R}{\otimes} a \mapsto a^{\langle 0\rangle} \underset{R}{\otimes} ha^{\langle 1\rangle} \tag{3.17}$$

gives rise to an entwining structure (A, C, ψ) over R. Hence the R-R bimodule $A \underset{R}{\otimes} H$ possesses an A-coring structure with left and right A-actions

$$a_1 \cdot (a \underset{R}{\otimes} h) \cdot a_2 = a_1 a a_2^{\langle 0\rangle} \underset{R}{\otimes} h a_2^{\langle 1\rangle} \qquad \text{for } a_1, a_2 \in A,\ a \underset{R}{\otimes} h \in A \underset{R}{\otimes} H,$$

coproduct $A \underset{R}{\otimes} \gamma$ and counit $A \underset{R}{\otimes} \pi$. This coring possesses a grouplike element $1_A \underset{R}{\otimes} 1_H$. The $\mathcal{H}_R$-extension $B \subset A$ was termed $\mathcal{H}_R$-Galois in [25] if the A-coring $A \underset{R}{\otimes} H$, associated to it above, is a Galois coring. This means bijectivity of the canonical map

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0\rangle} \underset{R}{\otimes} a'^{\langle 1\rangle}. \tag{3.18}$$

Analogously, in the case of a right comodule algebra A for the left bialgebroid $\mathcal{H}_L = (H, L, s, t, \gamma, \pi)$ the $\mathcal{H}_L$-Galois property of the extension $A^{co\mathcal{H}_L} \subset A$ means the bijectivity of the canonical map

$$\mathrm{can}_L : A \underset{A^{co\mathcal{H}_L}}{\otimes} A \to A \underset{L}{\otimes} H \qquad a \underset{A^{co\mathcal{H}_L}}{\otimes} a' \mapsto a_{\langle 0\rangle} a' \underset{L}{\otimes} a_{\langle 1\rangle}. \tag{3.19}$$

This is equivalent to the Galois property of the A-coring $A \underset{L}{\otimes} H$ with A-A bimodule structure

$$a_1 \cdot (a \underset{L}{\otimes} h) \cdot a_2 = a_{1\langle 0\rangle} a a_2 \underset{L}{\otimes} a_{1\langle 1\rangle} h \qquad \text{for } a_1, a_2 \in A,\ a \underset{L}{\otimes} h \in A \underset{L}{\otimes} H,$$

coproduct $A \underset{L}{\otimes} \gamma$, counit $A \underset{L}{\otimes} \pi$ and grouplike element $1_A \underset{L}{\otimes} 1_H$. (Recall from Section 2.2 that by our convention A is an L^{op}-ring in this case.)

Proposition 3.1 *This is withdrawn because of an unjustified step in the proof.*

Withdrawal of Proposition 3.1 means that – although we are not aware of any counterexample – it is *no longer proven* that for a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, any $\mathcal{H}_L$-comodule possesses an $\mathcal{H}_R$-comodule structure, or vice versa. (It is discussed in [7, (arXiv version) Theorem 2.5] under what additional assumptions this can be proven.) Therefore, Definition 3.2 and Lemma 3.3 need to be modified as follows. No other results in the rest of the paper are affected by these changes.

Definition 3.2 A right *comodule for the Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a triple (M, τ_L, τ_R), where the pair (M, τ_R) is a right $\mathcal{H}_R$-comodule and (M, τ_L) is a right $\mathcal{H}_L$-comodule such that the R-R and the L-L bimodule structures of M are related via

$$l \cdot m \cdot l' = \pi_R \circ t_L(l') \cdot m \cdot \pi_R \circ t_L(l), \tag{3.20}$$

and the compatibility relations

$$(\tau_R \underset{L}{\otimes} H) \circ \tau_L = (M \underset{R}{\otimes} \gamma_L) \circ \tau_R \tag{3.21}$$
$$(\tau_L \underset{R}{\otimes} H) \circ \tau_R = (M \underset{L}{\otimes} \gamma_R) \circ \tau_L \tag{3.22}$$

hold true.

The right $\mathcal{H}$-comodule (A, τ_L, τ_R) is said to be a right *$\mathcal{H}$-comodule algebra* if (A, τ_R) is a right $\mathcal{H}_R$-comodule algebra and (A, τ_L) is a right $\mathcal{H}_L$-comodule algebra. [2]

We follow the convention of using upper indices to denote the components of the coaction of a right bialgebroid and lower indices in the case of a left bialgebroid.

Lemma 3.3 *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with a bijective antipode and let A be a right $\mathcal{H}$-comodule algebra. Then the subalgebras of $\mathcal{H}_R$-coinvariants and of $\mathcal{H}_L$-coinvariants in A coincide. Moreover, denoting this coinvariant subalgebra by B, the canonical map*

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0\rangle} \underset{R}{\otimes} a'^{\langle 1\rangle}$$

is injective/surjective/bijective if and only if the canonical map

$$\mathrm{can}_L : A \underset{B}{\otimes} A \to A \underset{L}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto a_{\langle 0\rangle} a' \underset{L}{\otimes} a_{\langle 1\rangle}$$

is injective/surjective/bijective.

For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ with a bijective antipode, and an $\mathcal{H}$-comodule algebra A with $\mathcal{H}_R$-, equivalently, $\mathcal{H}_L$-coinvariant subalgebra B, we term the algebra extension $B \subset A$ an *$\mathcal{H}$-extension*.

In the case when both canonical maps in Lemma 3.3 are bijective, we say that $B \subset A$ is an *$\mathcal{H}$-Galois extension*.

Proof of Lemma 3.3: For any right $\mathcal{H}$-comodule (M, τ_L, τ_R), there exists an isomorphism

$$\Phi_M : M \underset{R}{\otimes} H \to M \underset{L}{\otimes} H \qquad m \underset{R}{\otimes} h \mapsto m_{\langle 0\rangle} \underset{L}{\otimes} m_{\langle 1\rangle} S(h)$$

with inverse

$$\Phi_M^{-1} : M \underset{L}{\otimes} H \to M \underset{R}{\otimes} H \qquad m \underset{L}{\otimes} h \mapsto m^{\langle 0\rangle} \underset{R}{\otimes} S^{-1}(h) m^{\langle 1\rangle}.$$

Since $\Phi_M(\tau_R(m)) = m \otimes_L 1_H$ and $\Phi_M(m \otimes_R 1_H) = \tau_L(m)$, it follows that in particular the $\mathcal{H}_R$-coinvariants and the $\mathcal{H}_L$-coinvariants of any $\mathcal{H}$-comodule algebra A coincide.

Using the $\mathcal{H}_R$-colinearity of τ_L, the Hopf algebroid axiom (2.11), the fact that the image of τ_L is in the Takeuchi product $A \times_L H$, the L^{op}-ring structure of A and the counitality of τ_L, one checks that for an $\mathcal{H}$-comodule algebra (A, τ_L, τ_R) the identity $\Phi_A \circ \mathrm{can}_R = \mathrm{can}_L$ holds true. ∎

Example 3.4 A k-bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ is an example both of a left- and of a right bialgebroid via the correspondence $\mathcal{B} := (H, k, \eta, \eta, \Delta, \epsilon)$. A Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ is an example of a Hopf algebroid via $\mathcal{H} := (\mathcal{B}, \mathcal{B}, S)$.

A right H-comodule (algebra) (A, τ) is an example of a right $\mathcal{B}$-comodule (algebra) and gives rise to an $\mathcal{H}$-comodule (algebra) via (A, τ, τ). The H-coinvariants are obviously the same as the $\mathcal{B}$-coinvariants and the extension $A^{coH} \subset A$ is H-Galois if and only if it is $\mathcal{B}$-Galois.

Example 3.5 A weak bialgebra ([29, 8]) has been shown to determine a left bialgebroid in ([24, 31, 36]). Weak comodule algebras ([3, 18, 19]) over a weak bialgebra have been shown in ([12], Proposition 3.9) to be equivalent to comodule algebras for the corresponding left bialgebroid.

Now just the same way as a weak bialgebra determines a left bialgebroid, it determines also a right bialgebroid, and a weak Hopf algebra determines a Hopf algebroid ([9], Example 4.8). A

[2] It is proven in the arXiv version and the Corrigendum of [7] that the category $\mathcal{M}^{\mathcal{H}}$ of right comodules of a Hopf algebroid $\mathcal{H}$ is a monoidal category, with strict monoidal forgetful functors to the comodule categories of the constituent bialgebroids. Consequently, an $\mathcal{H}$-comodule algebra is the same as a monoid in $\mathcal{M}^{\mathcal{H}}$.

weak comodule algebra for the weak bialgebra is equivalent also to a comodule algebra for the corresponding right bialgebroid, and a weak comodule algebra for a weak Hopf algebra is equivalent to a comodule algebra for the corresponding Hopf algebroid.

The coinvariants of a weak comodule algebra for a weak bialgebra are the same as its coinvariants for the corresponding left or right bialgebroid.

A weak bialgebra extension $B \subset A$ is a Galois extension in the sense of [19] if and only if $B \subset A$ is a Galois extension by the corresponding right bialgebroid. In the case of a weak Hopf algebra with a bijective antipode, this is equivalent also to the Galois property of $B \subset A$ as an extension by the corresponding left bialgebroid.

Example 3.6 The total algebra H of a right bialgebroid $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ is a right $\mathcal{H}_R$-comodule algebra via γ. We have $H^{co\mathcal{H}_R} = t(R)$.

The total algebra H of a left bialgebroid $\mathcal{H}_L = (H, L, s, t, \gamma, \pi)$ is a right $\mathcal{H}_L$-comodule algebra via γ and $H^{co\mathcal{H}_L} = s(L)$.

For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, the total algebra is then a right $\mathcal{H}$-comodule algebra and the canonical map

$$\mathrm{can}_R : {}^R H \otimes H_R \to H^R \otimes {}^R H \qquad h \underset{R^{op}}{\otimes} h' \mapsto hh'^{(1)} \underset{R}{\otimes} h'^{(2)}$$

is bijective with inverse

$$\mathrm{can}_R^{-1} : H^R \otimes {}^R H \to {}^R H \otimes H_R \qquad h \underset{R}{\otimes} h' \mapsto hS(h'_{(1)}) \underset{R^{op}}{\otimes} h'_{(2)}.$$

That is, the extension $t_R : R^{op} \to H$ is $\mathcal{H}_R$-Galois. If the antipode S is bijective then also the canonical map

$$\mathrm{can}_L : H^L \otimes {}_L H \to H_L \otimes {}_L H \qquad h \underset{L}{\otimes} h' \mapsto h_{(1)} h' \underset{L}{\otimes} h_{(2)}$$

is bijective with inverse

$$\mathrm{can}_L^{-1} : H_L \otimes {}_L H \to H^L \otimes {}_L H \qquad h \underset{L}{\otimes} h' \mapsto h'^{(2)} \underset{L}{\otimes} S^{-1}(h'^{(1)})h,$$

that is also the extension $s_L : L \to H$ is $\mathcal{H}_L$-Galois.

Example 3.7 Let $\mathcal{H}$ be a Hopf algebroid and A a right $\mathcal{H}_R$-module algebra. The smash product algebra [26] $A \rtimes H$ is defined as the k-module $A^R \otimes {}^R H$ with multiplication

$$(a \rtimes h)(a' \rtimes h') := a'(a \cdot h'^{(1)}) \rtimes hh'^{(2)}.$$

With this definition $A \rtimes H$ is an R-ring via the homomorphism

$$R \to A \rtimes H \qquad r \mapsto 1_A \rtimes s_R(r)$$

or an L^{op}-ring via the anti-homomorphism

$$L \to A \rtimes H \qquad l \mapsto 1_A \rtimes t_L(l).$$

One can introduce right $\mathcal{H}_L$- and right $\mathcal{H}_R$-comodule structures on $A \rtimes H$ via $\tau_L := A \underset{R}{\otimes} \gamma_L$ and $\tau_R := A \underset{R}{\otimes} \gamma_R$, respectively. The triple $(A \rtimes H, \tau_L, \tau_R)$ is a right $\mathcal{H}$-comodule algebra. We have $(A \rtimes H)^{co\mathcal{H}_R} = \{a \rtimes 1_H\}_{a \in A}$ and the canonical map

$$\begin{array}{rcl} \mathrm{can}_R : (A \rtimes H) \underset{A}{\otimes} (A \rtimes H) \simeq A^R \otimes {}^R H_{\mathbf{R}} \otimes {}^{\mathbf{R}} H & \to & (A \rtimes H) \underset{R}{\otimes} H \simeq A^R \otimes {}^R H^{\mathbf{R}} \otimes {}^{\mathbf{R}} H \\ a \underset{R}{\otimes} h \underset{\mathbf{R}}{\otimes} h' & \mapsto & a \underset{R}{\otimes} h' h^{(1)} \underset{\mathbf{R}}{\otimes} h^{(2)} \end{array}$$

is bijective with inverse

$$\mathrm{can}_R^{-1} : A^R \otimes {}^R H^{\mathbf{R}} \otimes {}^{\mathbf{R}} H \to A^R \otimes {}^R H_{\mathbf{R}} \otimes {}^{\mathbf{R}} H \qquad a \underset{R}{\otimes} h \underset{\mathbf{R}}{\otimes} h' \mapsto a \underset{R}{\otimes} h'_{(2)} \underset{\mathbf{R}}{\otimes} hS(h'_{(1)}).$$

This means that the extension $A \subset A \rtimes H$ is $\mathcal{H}_R$-Galois. If the antipode of $\mathcal{H}$ is bijective then it is also $\mathcal{H}_L$-Galois.

Example 3.8 In ([25], Theorem 5.1) Kadison has shown that a depth 2 extension $B \subset A$ of k-algebras – if it is balanced or faithfully flat – is a Galois extension for the right bialgebroid, constructed in [26] on the total algebra $(A \underset{B}{\otimes} A)^B$ (the centralizer of B in the canonical bimodule $A \underset{B}{\otimes} A$).

Recall that if the extension $B \subset A$ is in addition a Frobenius extension, $(A \underset{B}{\otimes} A)^B$ possesses a Frobenius Hopf algebroid structure [9]. Extending the result of [25] considerably, Bálint and Szlachányi have shown in ([2], Theorem 3.7) that an extension $B \subset A$ of k-algebras is $\mathcal{H}$-Galois for some Frobenius Hopf algebroid $\mathcal{H}$ if and only if it is a balanced depth 2 Frobenius extension.

4 A Kreimer-Takeuchi type theorem for Hopf algebroids

In this section we investigate $\mathcal{H}$-extensions for finitely generated projective Hopf algebroids $\mathcal{H}$ with a bijective antipode. We show that for an $\mathcal{H}$-Galois extension $B \subset A$, the algebra A is projective both as a left and as a right B-module and – under the additional assumption that $(A \underset{k}{\otimes} A)^{co\mathcal{H}} \simeq A \underset{k}{\otimes} B$, see below – the surjectivity of the canonical map (3.18) implies its bijectivity. This is a generalization of the classical theorem for finitely generated projective Hopf algebras by Kreimer and Takeuchi ([27], Theorem 1.7).

Recently Schauenburg and Schneider [33] have used new ideas to prove the Kreimer-Takeuchi theorem and generalizations of it. Their arguments are formulated in terms of entwining structures [13] over a commutative ring. In what follows we claim that the line of reasoning in [33] can be applied almost without modification to entwining structures over non-commutative algebras so to prove a Kreimer-Takeuchi type theorem for Hopf algebroids.

As we have seen at the beginning of Section 3, a right comodule algebra A for a right R-bialgebroid $\mathcal{H}_R$ determines an entwining structure (3.17) over R.

Lemma 4.1 *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode and A be a right $\mathcal{H}$-comodule algebra. The map ψ in (3.17), corresponding to the right $\mathcal{H}_R$-comodule algebra structure of A, is an isomorphism.*

Proof: The inverse of ψ is constructed using the $\mathcal{H}_L$-coaction $a \mapsto a_{\langle 0\rangle} \underset{L}{\otimes} a_{\langle 1\rangle}$ on A, as

$$\psi^{-1} : A \underset{R}{\otimes} H \to H \underset{R}{\otimes} A \qquad a \underset{R}{\otimes} h \mapsto hS^{-1}(a_{\langle 1\rangle}) \underset{R}{\otimes} a_{\langle 0\rangle}.$$

■

Motivated by Lemma 4.1, we study bijective entwining structures over R. We are going to generalize ([33], Theorem 3.1). Recall that for any entwined module M and any k-module V, the k-module $V \underset{k}{\otimes} M$ is an entwined module via the second tensor factor. The elements of $V \underset{k}{\otimes} M^{coC}$ form a subset of $(V \underset{k}{\otimes} M)^{coC}$. We have $V \underset{k}{\otimes} M^{coC} = (V \underset{k}{\otimes} M)^{coC}$ if, for example, the k-module V is flat.

Proposition 4.2 *Let (A, C, ψ) be a bijective entwining structure over the algebra R, such that the R-coring C possesses a grouplike element e. Denote the corresponding right C-coaction on A by $a_{\langle 0\rangle} \underset{R}{\otimes} a_{\langle 1\rangle} :\, = \psi(e \underset{R}{\otimes} a)$ and denote the subring of its coinvariants by B. Assume that C is flat as a left (right) R-module and projective as a right (left) C-comodule.*

(1) *Suppose that $(A \underset{k}{\otimes} A)^{coC} = A \underset{k}{\otimes} B$ and the canonical map*

$$\mathrm{can} : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} C \qquad a \underset{B}{\otimes} a' \mapsto aa'_{\langle 0\rangle} \underset{R}{\otimes} a'_{\langle 1\rangle} \tag{4.23}$$

is surjective. Under these assumptions the canonical map (4.23) is bijective.

(2) *If the canonical map (4.23) is bijective then A is projective as a right (left) B-module.*

Proof: The proof is actually the same as the proof of ([33], Theorem 3.1), so we present only a sketchy proof here.

(1) Let us use the assumption that C is projective as a right C-comodule. By the bijectivity of ψ and the adjunction (2.15), for any entwined module M we have $\mathrm{Hom}_A^C(A \underset{R}{\otimes} C, M) \simeq \mathrm{Hom}^C(C, M)$.

The forgetful functor $\mathcal{M}_A^C \to \mathcal{M}^C$ is a right adjoint, hence it preserves monomorphisms. A morphism in any of the (comodule) categories $\mathcal{M}_A^C$ and $\mathcal{M}^C$ is an epimorphism if and only if it is a surjective map, hence the forgetful functor $\mathcal{M}_A^C \to \mathcal{M}^C$ preserves also epimorphisms. Therefore, in light of ([16], 18.20(2)), by flatness of the left R-module C, projectivity of the right C-comodule C implies projectivity of the entwined module $A \underset{R}{\otimes} C$. Hence the surjective map

$$A \underset{k}{\otimes} A \to A \underset{R}{\otimes} C \qquad a \underset{k}{\otimes} a' \mapsto aa'_{\langle 0\rangle} \underset{R}{\otimes} a'_{\langle 1\rangle} \tag{4.24}$$

is a split epimorphism of entwined modules. This means that $A \underset{R}{\otimes} C$ is a direct summand of $A \underset{k}{\otimes} A$.

Notice that $(A \underset{R}{\otimes} C)^{coC} \simeq A$ via the isomorphism

$$\alpha : A \to (A \underset{R}{\otimes} C)^{coC} \qquad a \mapsto a \underset{R}{\otimes} e,$$

hence the canonical map (4.23) is related to the unit of the adjunction (2.16) as can $= \mu_{A \underset{R}{\otimes} C} \circ (\alpha \underset{B}{\otimes} A)$. This means that can is bijective provided $\mu_{A \underset{k}{\otimes} A}$ is bijective. Tensoring a k-free resolution

$$\dots P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \longrightarrow 0$$

of A with A over k, we can write $A \underset{k}{\otimes} A$ as the cokernel of the morphism $\partial_1 \underset{k}{\otimes} A : P_1 \underset{k}{\otimes} A \to P_0 \underset{k}{\otimes} A$ between entwined modules that are direct sums of copies of A. Functoriality of μ implies that

$$\mu_{A \underset{k}{\otimes} A} \circ [(\partial_0 \underset{k}{\otimes} B) \underset{B}{\otimes} A] = (\partial_0 \underset{k}{\otimes} A) \circ (P_0 \underset{k}{\otimes} \mu_A).$$

Since μ_A is an isomorphism, we can use the universality of the cokernel to define the inverse of $\mu_{A \underset{k}{\otimes} A}$ as the unique map $\phi : A \underset{k}{\otimes} A \to (A \underset{k}{\otimes} A)^{coC} \underset{B}{\otimes} A = (A \underset{k}{\otimes} B) \underset{B}{\otimes} A$ which satisfies

$$\phi \circ (\partial_0 \underset{k}{\otimes} A) = [(\partial_0 \underset{k}{\otimes} B) \underset{B}{\otimes} A] \circ (P_0 \underset{k}{\otimes} \mu_A^{-1}).$$

This proves the bijectivity of $\mu_{A \underset{k}{\otimes} A}$, hence of the canonical map (4.23).

If C is projective as a left C-comodule and flat as a right R-module then ψ has to be replaced with its inverse in the above line of reasoning.

(2) Projectivity of the right B-module A is proven from the projectivity of the right C-comodule C as follows. By bijectivity of the canonical map (4.23) and using the adjunction (2.16), for any entwined module M we have $\mathrm{Hom}^C(C, M) \simeq \mathrm{Hom}_B(A, M^{coC})$. Hence the functor $\mathrm{Hom}_B(A, (\ \_\ )^{coC}) : \mathcal{M}_A^C \to \mathcal{M}_k$ is exact.

Let $f : \oplus_I B \to A$ be an epimorphism in $\mathcal{M}_B$, for some index set I. Then $f \underset{B}{\otimes} A$ is an epimorphism in $\mathcal{M}_A^C$ which is mapped by the functor $\mathrm{Hom}_B(A, (\ \_\ )^{coC})$ to the epimorphism $\mathrm{Hom}_B(A, f)$ in $\mathcal{M}_k$.

In order to prove projectivity of the left B-module A from projectivity of the left C-comodule C, replace ψ by its inverse in the above arguments. ■

As it has been explained to us by Tomasz Brzeziński, there exists an alternative, more general argument proving bijectivity of the canonical map (4.23) from split surjectivity of (4.24) in $\mathcal{M}_A^C$. Namely, application of ([15], Theorem 2.1) to the A-coring $A \underset{R}{\otimes} C$, corresponding to the R-entwining structure (A, C, ψ) with grouplike element $e \in C$, and its comodule $M = A$, implies the claim. Indeed, in this case condition b) of ([15], Theorem 2.1) reduces to $(A \underset{k}{\otimes} A)^{coC} \simeq A \underset{k}{\otimes} A^{coC}$.

We have formulated Proposition 4.2 (1) in terms of the assumption $(A \underset{k}{\otimes} A)^{coC} = A \underset{k}{\otimes} A^{coC}$. It has the advantage that in certain situations (e.g. if k is a field) it obviously holds true. We do not know, however, whether it is also a necessary condition for the claim of Proposition 4.2 (1). On the other hand, notice that if the canonical map (4.23) is an isomorphism in $\mathcal{M}_A^C$ then

$$(A \underset{B}{\otimes} A)^{coC} \simeq (A \underset{R}{\otimes} C)^{coC} = A, \tag{4.25}$$

where B stands for A^{coC}, as before. Application of ([40], Proposition 5.1) to the A-coring $A \underset{R}{\otimes} C$, corresponding to the R-entwining structure (A, C, ψ) with grouplike element $e \in C$, and its

comodule $M = A$, implies that the bijectivity of the canonical map (4.23) follows from split surjectivity of (4.24) also under the (sufficient and necessary) condition (4.25).

Let us turn to the application of Proposition 4.2 to Hopf algebroid extensions. Let $\mathcal{H}$ be a finitely generated projective Hopf algebroid with a bijective antipode. It follows from the Fundamental Theorem for Hopf algebroids ([6], Theorem 4.2, see also the *Corrigendum*) and the existence of a Hopf module structure on $H^* = \mathrm{Hom}_R(H,R)$ with coinvariants $\mathcal{L}(H^*)$ ([6], Proposition 4.4), that for such a Hopf algebroid the map

$$\alpha_L : {}^L H \otimes \mathcal{L}(H^*)^L \ \to \ H^* \qquad h \underset{L^{op}}{\otimes} \lambda^* \ \mapsto \lambda^* \leftharpoonup S(h) \equiv \lambda^*[S(h)\,\_\,] \tag{4.26}$$

is an isomorphism of Hopf modules, hence in particular of left H-modules. (The left H-module structure on ${}^L H \otimes \mathcal{L}(H^*)^L$ is given by left multiplication in the first tensor factor, and on H^* by $h \cdot \phi^* : = \phi^* \leftharpoonup S(h) \equiv \phi^*[S(h)\,\_\,]$.) This implies that the element $\alpha_L^{-1}(\pi_R)$ is an invariant of the left H-module ${}^L H \otimes \mathcal{L}(H^*)^L$. The elements $\{x_i\} \subset H$ and $\{\lambda_i^*\} \subset \mathcal{L}(H^*)$ satisfying $\sum_i x_i \underset{L^{op}}{\otimes} \lambda_i^* = \alpha_L^{-1}(\pi_R)$ can be used to construct dual bases for the left ${}^*H$-module on H defined as ${}^*\phi \rightharpoondown h : = h^{(1)}\ s_R \circ {}^*\phi(h^{(2)})$. As a matter of fact, for $\lambda^* \in \mathcal{L}(H^*)$ we have $\lambda^* \circ S \in \mathcal{L}({}^*H)$ ([6], Scholium 2.10), and ${}^*H$ is a right H module via ${}^*\phi \leftharpoondown h = {}^*\phi(h\,\_\,)$. We leave it to the reader to check that the sets $\{x_i\} \subset H$ and $\{\lambda_i^* \circ S \leftharpoondown S^{-1}(\,\_\,)\} \subset {}_{{}^*H}\mathrm{Hom}(H, {}^*H)$ are dual bases, showing that H is a finitely generated and projective ${}^*H$-module.

Since the antipode was assumed to be bijective, we can apply the same argument to the co-opposite Hopf algebroid $\mathcal{H}_{cop}$ to conclude on the finitely generated projectivity of the left H^*-module on H defined as $\phi^* \rightharpoonup h : = h^{(2)}\ t_R \circ \phi^*(h^{(1)})$.

Furthermore, projectivity of H as a left ${}^*H$ module implies that it is projective as a right $\mathcal{H}_R$-comodule and projectivity of H as a left H^*-module implies that it is projective as a left $\mathcal{H}_R$-comodule. These observations allow for the application of Proposition 4.2 to the entwining structure (3.17) – which is bijective by Lemma 4.1.

Corollary 4.3 *Let $\mathcal{H}$ be a finitely generated projective Hopf algebroid with a bijective antipode and let $B \subset A$ be an $\mathcal{H}$-extension.*

(1) *Suppose that $(A \underset{k}{\otimes} A)^{co\mathcal{H}} = A \underset{k}{\otimes} B$ (e.g. A is k-flat). If the canonical map*

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0\rangle} \underset{R}{\otimes} a'^{\langle 1\rangle}$$

is surjective then the extension $B \subset A$ is $\mathcal{H}_R$-Galois (equivalently, $\mathcal{H}_L$-Galois).

(2) *If the extension $B \subset A$ is $\mathcal{H}$-Galois then A is projective both as a left and as a right B-module.*

5 Morita theory for Hopf algebroid extensions

As we have seen at the beginning of Section 3, an $\mathcal{H}_R$-extension $B \subset A$, for a right R -bialgebroid $\mathcal{H}_R$, determines an A-coring structure on $A \underset{R}{\otimes} H$. One can apply the Morita theory for corings, developed in [20], to this coring. In particular, if $\mathcal{H}_R$ is finitely generated and projective as a left R-module, then Theorem 2.5 (1) can be used to obtain criteria which are equivalent to the $\mathcal{H}_R$-Galois property of the extension $B \subset A$ and the projectivity of the left B-module A. Analogously, one can apply Theorem 2.5 (1) to obtain criteria which are equivalent to the Galois property of an $\mathcal{H}_L$-extension $B \subset A$ for a finitely generated projective left bialgebroid $\mathcal{H}_L$ and the projectivity of the right B-module A.

Applying the results of the previous sections, we prove that if $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a finitely generated projective Hopf algebroid with a bijective antipode and A is a right $\mathcal{H}$-comodule algebra, with $\mathcal{H}_R$, equivalently, $\mathcal{H}_L$-coinvariant subalgebra B, then the equivalent conditions, derived from Theorem 2.5 (1) for $B \subset A$ as an $\mathcal{H}_R$-extension, on one hand, and as an $\mathcal{H}_L$-extension, on the other hand, are equivalent also to each other.

Let $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ be a right bialgebroid such that H is finitely generated and projective as a left R-module and let A be a right $\mathcal{H}_R$-comodule algebra. As a first step, let us identify the

Morita context $(A^{*C}, {}^*C, A, {}^*C^{*C}, \nu, \mu)$, associated to the A-coring $C = A \underset{R}{\otimes} H$ (as it is explained in Section 2.4).

Recall that A – being a right $\mathcal{H}_R$-comodule – is also a left ${}^*H$-module via ${}^*\phi \cdot a = a^{\langle 0\rangle} \cdot {}^*\phi(a^{\langle 1\rangle})$. Since the module ${}^RH$ is finitely generated and projective, ${}^*H$ possesses a left bialgebroid structure. Let us introduce the smash product algebra ${}^*H \ltimes A$ as the k-module ${}^*H \underset{R}{\otimes} A$ (where the right R-module structure of ${}^*H$ is given by $({}^*\phi \cdot r)(h) = {}^*\phi(h)r$) with multiplication

$$({}^*\phi \ltimes a)({}^*\psi \ltimes a'){:}= {}^*\psi_{(1)}{}^*\phi \ltimes ({}^*\psi_{(2)} \cdot a)a'. \tag{5.27}$$

With this definition ${}^*H \ltimes A$ is an A-ring via the homomorphism

$$i_A : A \to {}^*H \ltimes A \qquad a \mapsto \pi \ltimes a.$$

Define a right-right *relative* $(A, \mathcal{H}_R)$ *-module* to be an entwined module for the R -entwining structure (3.17), i.e. a right comodule for the A-coring $A \underset{R}{\otimes} H$. This means a right A-module (and hence in particular a right R-module) and a right $\mathcal{H}_R$-comodule M such that the compatibility condition

$$(m \cdot a)^{\langle 0\rangle} \underset{R}{\otimes} (m \cdot a)^{\langle 1\rangle} = m^{\langle 0\rangle} \cdot a^{\langle 0\rangle} \underset{R}{\otimes} m^{\langle 1\rangle} a^{\langle 1\rangle}$$

holds true for any $m \in M$ and $a \in A$. Clearly, A is itself a relative $(A, \mathcal{H}_R)$-module. It is straightforward to check that the category $\mathcal{M}_A^H$ of relative $(A, \mathcal{H}_R)$-modules is isomorphic to the category $\mathcal{M}_{{}^*H \ltimes A}$ of right ${}^*H \ltimes A$-modules and the ${}^*H \ltimes A$-invariants are the same as the $\mathcal{H}_R$-coinvariants. This fact can be easily understood in view of

Lemma 5.1 *Let $\mathcal{H}_R$ be a right R-bialgebroid such that H is finitely generated and projective as a left R-module and let A be a right $\mathcal{H}_R$-comodule algebra. Then the left A-dual algebra of the A-coring $A \underset{R}{\otimes} H$ is isomorphic to the smash product algebra ${}^*H \ltimes A$.*

Proof: The required algebra isomorphism is constructed as

$${}^*H \ltimes A \to {}_A\mathrm{Hom}(A \underset{R}{\otimes} H, A) \qquad {}^*\phi \ltimes a \mapsto (\ a' \underset{R}{\otimes} h \mapsto a'({}^*\phi(h) \cdot a)\ ).$$

■

The Morita context, associated to the $\mathcal{H}_R$-extension $B \subset A$, is then $(B, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}_R}$ $\nu_R, \mu_R)$ with connecting maps

$$\nu_R : A \underset{{}^*H\ltimes A}{\otimes} ({}^*H \ltimes A)^{co\mathcal{H}_R} \to B \qquad a' \underset{{}^*H\ltimes A}{\otimes} (\sum_i {}^*\phi_i \ltimes a_i) \mapsto \sum_i ({}^*\phi_i \cdot a')a_i \tag{5.28}$$

$$\mu_R : ({}^*H \ltimes A)^{co\mathcal{H}_R} \underset{B}{\otimes} A \to {}^*H \ltimes A \qquad (\sum_i {}^*\phi_i \ltimes a_i) \underset{B}{\otimes} a' \mapsto \sum_i {}^*\phi_i \ltimes a_i a'. \tag{5.29}$$

Since H is finitely generated and projective as a left R-module, so is the left A-module $A \underset{R}{\otimes} H$. Hence part (1) in Theorem 2.5 implies

Proposition 5.2 *Let $\mathcal{H}_R$ be a right R-bialgebroid such that H is finitely generated and projective as a left R-module and let $B \subset A$ be an $\mathcal{H}_R$-extension. The following assertions are equivalent.*

(a) *The map μ_R in (5.29) is surjective (and, a fortiori, bijective).*

(b) *The functor $(\ \_\ )^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is fully faithful.*

(c) *A is a right ${}^*H \ltimes A$-generator.*

(d) *A is projective as a left B-module and the map*

$${}^*H \ltimes A \to {}_B\mathrm{End}(A) \qquad {}^*\phi \ltimes a \mapsto (\ a' \mapsto ({}^*\phi \cdot a')a\ ) \tag{5.30}$$

is an algebra anti-isomorphism.

(e) *A is projective as a left B-module and the extension $B \subset A$ is $\mathcal{H}_R$-Galois.*

The arguments leading to Theorem 5.2 can be repeated by replacing the right bialgebroid $\mathcal{H}_R$ with a left L-bialgebroid $\mathcal{H}_L$ such that H is finitely generated and projective as a left L-module. Indeed, in this case the left L-dual, ${}_*H$, possesses a right bialgebroid structure and A is a right ${}_*H$-module via $a \cdot {}_*\phi = a_{\langle 0\rangle} \cdot {}_*\phi(a_{\langle 1\rangle})$. The right A-dual of the A-coring $A \underset{L}{\otimes} H$ is isomorphic to the smash product algebra ${}_*H \ltimes A$, which is defined as the k-module ${}_*H \underset{L}{\otimes} A$, (where the right L-module structure on ${}_*H$ is given by $({}_*\phi \cdot l)(h) = {}_*\phi(h)l$), with multiplication

$$({}_*\phi \ltimes a)({}_*\psi \ltimes a') \colon = {}_*\psi {}_*\phi^{(1)} \ltimes a(a' \cdot {}_*\phi^{(2)}).$$

Since A is an L^{op}-ring, a left A-module is in particular a right L-module and we have the isomorphism $(A \underset{L}{\otimes} H) \underset{A}{\otimes} M \simeq M \underset{L}{\otimes} H$ for any left A-module M. A left comodule for the A-coring $A \underset{L}{\otimes} H$ is then equivalent to a left A-module (hence in particular a right L-module) and a right $\mathcal{H}_L$-comodule M such that the compatibility condition

$$(a \cdot m)_{\langle 0\rangle} \underset{L}{\otimes} (a \cdot m)_{\langle 1\rangle} = a_{\langle 0\rangle} \cdot m_{\langle 0\rangle} \underset{L}{\otimes} a_{\langle 1\rangle} m_{\langle 1\rangle}$$

holds true for $m \in M$ and $a \in A$. Such modules are called left-right relative $(A, \mathcal{H}_L)$-modules and their category is denoted by ${}_A\mathcal{M}^H$. It follows that the category ${}_A\mathcal{M}^H$ is isomorphic also to ${}_{{}_*H \ltimes A}\mathcal{M}$, the category of left ${}_*H \ltimes A$-modules. The Morita context associated to the $\mathcal{H}_L$-extension $B \subset A$ is $(B, {}_*H \ltimes A, A, ({}_*H \ltimes A)^{co\mathcal{H}_L}, \nu_L, \mu_L)$ with connecting maps

$$\nu_L : ({}_*H \ltimes A)^{co\mathcal{H}_L} \underset{{}_*H \ltimes A}{\otimes} A \to B \qquad (\sum_i {}_*\phi_i \ltimes a_i) \underset{{}_*H \ltimes A}{\otimes} a' \mapsto \sum_i a_i(a' \cdot {}_*\phi_i) \tag{5.31}$$

$$\mu_L : A \underset{B}{\otimes} ({}_*H \ltimes A)^{co\mathcal{H}_L} \to {}_*H \ltimes A \qquad a' \underset{B}{\otimes} (\sum_i {}_*\phi_i \ltimes a_i) \mapsto \sum_i {}_*\phi_i \ltimes a'a_i. \tag{5.32}$$

Part (1) of Theorem 2.5 implies

Proposition 5.3 *Let $\mathcal{H}_L$ be a left L-bialgebroid such that H is finitely generated and projective as a left L-module and let $B \subset A$ be an $\mathcal{H}_L$-extension. The following assertions are equivalent.*

(a) *The map μ_L in (5.32) is surjective (and, a fortiori, bijective).*
(b) *The functor $(\ \_\ )^{co\mathcal{H}_L} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ is fully faithful.*
(c) *A is a left ${}_*H \ltimes A$-generator.*
(d) *A is projective as a right B-module and the map*

$${}_*H \ltimes A \to \mathrm{End}_B(A) \qquad {}_*\phi \ltimes a \mapsto (\ a' \mapsto a(a' \cdot {}_*\phi)\ ) \tag{5.33}$$

is an algebra isomorphism.
(e) *A is projective as a right B-module and the extension $B \subset A$ is $\mathcal{H}_L$-Galois.*

Combining Proposition 5.2, 5.3, Corollary 4.3 and Lemma 3.3 we can state our main result.

Theorem 5.4 *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a finitely generated projective Hopf algebroid with a bijective antipode and let $B \subset A$ be an $\mathcal{H}$-extension. The following assertions are equivalent.*

(a) *The extension $B \subset A$ is $\mathcal{H}_R$ -Galois.*
(b) *A is projective as a left B-module and the map (5.30) is an algebra anti-isomorphism.*
(c) *A is a right ${}^*H \ltimes A$-generator.*
(d) *The functor $(\ \_\ )^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is fully faithful.*
(e) *The map μ_R in (5.29) is surjective (and, a fortiori, bijective).*
(a′) *The extension $B \subset A$ is $\mathcal{H}_L$ -Galois.*
(b′) *A is projective as a right B-module and the map (5.33) is an algebra isomorphism.*
(c′) *A is a left ${}_*H \ltimes A$-generator.*
(d′) *The functor $(\ \_\ )^{co\mathcal{H}_L} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ is fully faithful.*
(e′) *The map μ_L in (5.32) is surjective (and, a fortiori, bijective).*

Proof: It follows from part (2) of Corollary 4.3 that (a) is equivalent to any of the assertions

$$A \text{ is projective as a left } B \text{ module and the extension } B \subset A \text{ is } \mathcal{H}_R \text{ -Galois.} \tag{5.34}$$
$$A \text{ is projective as a right } B \text{ module and the extension } B \subset A \text{ is } \mathcal{H}_R \text{ -Galois.} \tag{5.35}$$

By Lemma 3.3 assertions (a) and (a′) are equivalent and (5.35) is equivalent to

$$A \text{ is projective as a right } B \text{ module and the extension } B \subset A \text{ is } \mathcal{H}_L \text{ -Galois.} \tag{5.36}$$

The rest of the proof follows from Propositions 5.2 and 5.3. ■

Let us mention that in ([17], Theorem 4.7) a stronger version of Theorem 2.5 (1) has been proven. Its application to bialgebroid extensions implies

Proposition 5.5 *Let $\mathcal{H}_R$ be a right R- bialgebroid such that H is finitely generated and projective as a left R-module and let $B \subset A$ be an $\mathcal{H}_R$-extension. The following assertions are equivalent.*
(a) *The Morita context $(B, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}_R}, \nu_R, \mu_R)$ is strict.*
(b) *The functor $(\ \_\ )^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is an equivalence with inverse $\_ \underset{B}{\otimes} A : \mathcal{M}_B \to \mathcal{M}_A^H$.*
(c) *The map (5.30) is an algebra anti-isomorphism and A is a left B-progenerator.*
(d) *A is faithfully flat as a left B-module and the extension $B \subset A$ is $\mathcal{H}_R$-Galois.*

Also the analogue of Proposition 5.5 for left bialgebroid extensions can be proven. We can not prove, however, that for an $\mathcal{H}$-extension, in the case of any finitely generated projective Hopf algebroid $\mathcal{H}$ with a bijective antipode, the equivalent conditions in Proposition 5.5 and their counterparts on the left bialgebroid of $\mathcal{H}$ are equivalent also to each other (as it was seen to be the case with Proposition 5.2 and 5.3). [3]

The Morita context $(A^{{}^*C}, {}^*C, A, {}^*C^{{}^*C}, \nu, \mu)$, associated in [20] to an A-coring C with a group-like element, is a generalization of the Morita context, associated to a bialgebra extension in [23]. In the case of a finite dimensional Hopf algebra over a field (or a Frobenius Hopf algebra over a commutative ring) another Morita context has been associated to a Hopf algebra extension in [22, 21]. The relation of the two Morita contexts is of the type described in part (2) of Theorem 2.5. In order to see what is the analogue of the Morita context of Cohen, Fishman and Montgomery in the case of Hopf algebroids, in the rest of the section we assume that $\mathcal{H}$ is a *Frobenius* Hopf algebroid.

Lemma 5.6 *Let $\mathcal{H}$ be a Frobenius Hopf algebroid and A be a left $\mathcal{H}_L$-module algebra. Consider the smash product algebra $H \ltimes A$, which is the k-module $H \underset{L}{\otimes} A$ with multiplication*

$$(h \ltimes a)(g \ltimes a') : = g_{(1)} h \ltimes (g_{(2)} \cdot a)a'.$$

The extension

$$i : A \to H \ltimes A \qquad a \mapsto 1_H \ltimes a$$

is a Frobenius extension.

Proof: Recall (from Section 2.1) that a Frobenius Hopf algebroid possesses non-degenerate left integrals. Let us fix such an integral ℓ and denote by ρ_* the unique element in H_*, for which $\ell \leftharpoonup \rho_* \equiv s_L \circ \rho_*(\ell_{(1)})\ell_{(2)} = 1_H$. A Frobenius functional $\Phi : H \ltimes A \to A$ is given by $h \ltimes a \mapsto \rho_*(h) \cdot a$. A Hopf algebroid calculation shows that it is an A-A bimodule map and possesses a dual basis $(S(\ell^{(2)}) \ltimes 1_A) \underset{A}{\otimes} (\ell^{(1)} \ltimes 1_A)$. ■

Recall that for a Frobenius Hopf algebroid $\mathcal{H}$ also the left bialgebroid ${}^*H$ possesses a Frobenius Hopf algebroid structure. Applying Lemma 5.6 together with part (2) of Theorem 2.5 we conclude

[3] This question is answered in [1], see Proposition 16 in the *Corrigendum* or Proposition 4.4 in the arXiv version.

that the Morita context $(A^{co\mathcal{H}}, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}}, \nu_R, \mu_R)$, associated to the right $\mathcal{H}_R$ -comodule algebra structure of a right $\mathcal{H}$-comodule algebra A for the Frobenius Hopf algebroid $\mathcal{H}$, is equivalent to the Morita context $(A^{co\mathcal{H}}, {}^*H \ltimes A, A, A, \nu', \mu')$ with connecting maps

$$\nu' : A \underset{{}^*H\ltimes A}{\otimes} A \to A^{co\mathcal{H}} \qquad a \underset{{}^*H\ltimes A}{\otimes} a' \mapsto {}^*\lambda \cdot (aa') \tag{5.37}$$

$$\mu' : A \underset{A^{co\mathcal{H}}}{\otimes} A \to {}^*H \ltimes A \qquad a \underset{A^{co\mathcal{H}}}{\otimes} a' \mapsto (\pi_R \ltimes a)({}^*\lambda \ltimes 1_A)(\pi_R \ltimes a'), \tag{5.38}$$

where ${}^*\lambda$ is a non-degenerate left integral in ${}^*H$.

Corollary 5.7 *If we add to the conditions of Theorem 5.4 the requirement that $\mathcal{H}$ be a Frobenius Hopf algebroid then we can add to the equivalent assertions (a)-(e′) also*

(f) For any non-degenerate left integral ${}^*\lambda$ in the Frobenius Hopf algebroid ${}^*H$ the map

$$A \underset{B}{\otimes} A \to {}^*H \ltimes A \qquad a \underset{B}{\otimes} a' \mapsto (\pi_R \ltimes a)({}^*\lambda \ltimes 1_A)(\pi_R \ltimes a')$$

is surjective (and, a fortiori, bijective).

By ([9], Lemma 5.14), for any non-degenerate left integral ℓ in a Hopf algebroid $\mathcal{H}$, $\rho_* : = \ell_L^{-1}(1_H) \in H_*$, and any element h of H, we have $\ell_{(1)}h \underset{L}{\otimes} \ell_{(2)} = \ell_{(1)} \underset{L}{\otimes} \ell_{(2)}t_L \circ \rho_*(\ell h^{(1)})S(h^{(2)})$. This implies that the image of the map (5.38) is an ideal in ${}^*H \ltimes A$. Hence – just as it has been proven for finite dimensional Hopf algebras in ([22], Corollary 1.3) – we see that it is true also for $\mathcal{H}$-extensions $B \subset A$ for a Frobenius Hopf algebroid $\mathcal{H}$ that if the k-algebra ${}^*H \ltimes A$ is simple then the extension $B \subset A$ is $\mathcal{H}$-Galois.

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